Lévy flights in nonhomogeneous media: Distributed-order fractional equation approach

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A jumping process, defined in terms of the Lévi distributed jumping size and the Poissonian, position-dependent waiting time with the algebraic jumping rate, is discussed on the assumption that parameters of both distributions are themselves random variables which are determined from given probability distributions. The fractional equation for the distributed Lévy order parameter μ is derived and solved. The solution is of the form of a combination of the Fox functions and simple scaling is lacking. The problem of accelerated diffusion is also discussed. The case of the distributed waiting time parameter θ is similarly solved and the solution offers a possibility to manage processes which are characterized by more general forms of the jumping rate, not only algebraic. Moreover, we mention a possibility that the parameters μ and θ are mutually dependent.

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I. INTRODUCTION

The continuous-time random walk (CTRW) [1,2] is a useful formalism to deal with the anomalous transport. It involves two quantities: a jump-length probability distribution $\lambda(x)$ and a waiting-time distribution $w(t)$ which refers to the time lapse between jumps. The Poissonian form of $w(t)$ means that the process is Markovian and it corresponds to the Brownian motion. If $w(t)$ has long tails and divergent moments, i.e., when trajectories are subjected to long rests, the motion appears subdiffusive [3]. As regards the jump-size distribution $\lambda(x)$, the stable forms are of particular interest. The stability is understood here in a sense of the central limit theorem: in the framework of the renormalization group method one can demonstrate that the stable distribution is reached after many steps of the convolution [4]. Often it is not sufficient to assume the standard Gaussian and the general Lévy distribution should be considered. In the latter case the trajectories exhibit Lévy flights with divergent moments and then the diffusion is accelerated. One can expect the presence of long jumps in systems of high complexity which are characterized by long-range correlations and nonlocal interactions. It is the case in biological systems where the presence of the Lévy statistics is attributed to fractals [5]. One can encounter long jumps in many social and environmental problems which are related to people mobility [6], e.g., to the spreading of infectious diseases [7,8]. In all those problems the medium has the nontrivial structure and then its nonuniformity should be included in the models. Since the tails of the Lévy distribution are powerlaw, $\sim|x|^{-1-\mu}(0<\mu<2)$, the infinite jumps are probable which generally can contradict physical requirements. Then one needs to introduce a cutoff to take into account a finite size of the system. An additional modification is necessary if the trajectory refers to a motion of a massive particle in the configurational space (Lévy walks). However if the time is sufficiently long, trajectories of the Lévy walks resemble Lévy flights where each jump length is distributed in the Lévy stable fashion [9].

The Lévy walk model introduces a coupling between the distributions $\lambda(x)$ and $w(t)$. We get another form the coupled CTRW by assuming $w(t)$ in the Poissonian form with the x-dependent jumping frequency $\nu(x)$, $w(t)=\nu(x)\exp[-\nu(x)t]$ [10]. For $\nu(x)=|x|^{-\theta}$, where $\theta>-1$, one can observe the anomalous diffusion [11] though the Markovian property holds in this case. The above form of $\nu(x)$ takes into account that the temporal properties of the process can change with the position and, in particular, it is well suited to handle the transport in fractals [12–14]. Moreover, the power-law form of the diffusion coefficient has been used to describe, e.g., the transport of fast electrons in a hot plasma [15] and the turbulent two-particle diffusion [16]. In the following, we assume that the jumping-size distribution $\lambda(x)$ has the Lévy shape. In the diffusion limit of small wave numbers, the master equation for the process which is characterized by the x-dependent jumping rate and the Lévy distributed jumping size, takes the form of the following fractional equation:

$$\frac{\partial p(x,t)}{\partial t}=K^{\mu}\frac{\partial^{\mu}[\nu(x)p(x,t)]}{\partial|x|^{\mu}}, \tag{1}$$

where $1<\mu<2$. The solution can be expressed in form of the Fox function [17,18]

$$p(x,t)=\mathcal{N}aH_{2,2}^{1,1}\left[a|x|\left|\begin{matrix}\left(1-\frac{1-\theta}{\mu+\theta},\frac{1}{\mu+\theta}\right),\left(1-\frac{1-\theta}{2+\theta},\frac{1}{2+\theta}\right)\\(\theta,1),\left(1-\frac{1-\theta}{2+\theta},\frac{1}{2+\theta}\right)\end{matrix}\right.\right], \tag{2}$$

where $a \sim t^{-1/(\mu+\theta)}$ [11,19]. The second moment of the distribution (2) diverges because its asymptotics falls slowly: $p(x,t) \sim |x|^{-\mu-1}(|x| \to \infty)$, where the proportionality coefficient depends on θ. The diffusion is then accelerated for any θ. In order to quantify the diffusion rate, one can evaluate moments of the order δ for $\delta < \mu$. Note that the asymptotic shape of $p(x,t)$ is determined solely by the parameter μ because of its power law form. The parameter θ, in turn, influences the slope and then it enters the formula for the fractional moments. One can classify the accelerated diffusion process—in terms of the fractional diffusion coefficient—as normal diffusion, subdiffusion, or superdiffusion, if θ is zero, positive, or negative, respectively [11].

The form of the distribution (2) indicates the self-similarity in a sense that $p(x,t)$ at an arbitrary time t can be expressed by scaling of the x variable

$$x(t) = t^{1/(\mu+\theta)}x(1) \equiv t^H x(1), \tag{3}$$

where H is the Hurst exponent [20]. This form of scaling is typical for fractals and can be encountered, accompanied by the Lévy statistics, in many natural processes. A well-known example from the field of biology is the heart rate spectrum [5]. However, the scale may not be unique: the direct coupling between fluctuations at different scales can take place and only one value of the Hurst exponent is not sufficient to characterize a system. Such system which involves the entire spectrum of the fractal dimension is known as multifractal. The lack of simple scaling in the multifractals requires a generalization of Eq. (1) by admitting a spectrum of the Lévy index μ.

Multifractal structures can be recognized in various systems. They include geophysical and atmospheric phenomena [21,22], as well as financial markets (see Ref. [23], and references therein). Multifractality is of particular interest in biology and medicine: it emerges in the DNA sequences [24] and neuron spiking [25]. It has been established that both heartbeat interval [26] and cerebral blood flow signals [27] exhibit the multifractal structure. The spectrum of the fractal index narrows for patients with so different diseases as migraine and congestive heart failure, compared to healthy individuals.

The multiscaling is crucial for disorder systems at random critical points [28]. Also the turbulence involves many scales both in space and time and multiscale correlations are important [29]. In the Lévy walk approach to the turbulence [30]—which introduces an associating a time scale with jump distances—the multiplicity of scales is explicitly taken into account since it is formulated in terms of an integral equation which contains a coupled memory kernel.

In the field of solid state physics, the fractional equations formalism has been applied to microporous materials in order to describe particle diffusion within very narrow channels (single-file systems) [31]. Though those results agree with simulations and experiments for short times, the model fails for longer times. The authors of Ref. [31] conclude that the order parameter may be in fact time dependent and the fractional diffusion equations of distributed order can be the starting point for future developments of the single-file diffusion theory. The problem of diffusion in the microporous materials is physically important; the single-file diffusion is encountered in such phenomena as: ion transport in biological membranes [32], colloids in polymer solution [33], Markov chains in statistics [34], microfluidic devices [35], traffic flow [36], and molecules in zeolites [37]. Another field of possible application of fractional equations of the distributed order is the financial market. Distributions of financial data usually possess fast falling power-law tails. One can describe those processes in terms of the fractional equation with a truncated jump-size distribution, which is a special case of the equation with the distributed Lévy index [38].

In respect to the random walk, the multiscaling can enter the decoupled CTRW models in two ways: as the distributed order Lévy parameter μ and the distributed order of the fractional time derivative in the fractional non-Markovian diffusion equation [39–42]. In this paper we study fractional equations which correspond to the Markovian process, characterized by the x-dependent waiting time distribution. Therefore the model parameters, either μ or θ, are random variables. The former case is considered in Sec. II, where the Lévy parameter μ is assumed to be distributed according to a given function. Section III is devoted to the problem of the random jumping rate parameter θ and a possible mutual dependence between μ and θ is also mentioned.

II. FRACTIONAL EQUATION WITH THE DISTRIBUTED LÉVY PARAMETER

We consider the random walk process defined by the waiting time probability distribution $w(t)$ and the jump-size distribution $\lambda(x|\mu)$. They are of the form

$$w(t) = \nu(x,\theta)e^{-\nu(x,\theta)t}, \tag{4}$$

where $\nu(x,\theta) = |x|^{-\theta}$ $(\theta > -1)$, and

$$\lambda(x|\mu) = \sqrt{2/\pi}\int_0^\infty \exp(-K^\mu k^\mu)\cos(kx)dk, \tag{5}$$

respectively. The latter expression corresponds to the symmetric Lévy distribution. In addition we assume that the Lévy index μ is a stochastic variable and it is governed by the normalized distribution $f(\mu)$. The stationary transition probability for infinitesimal time intervals Δt consists of two terms: the probability that no jump occurs during that time and that exactly one jump occurs. It reads

$$p_{tr}(x,\Delta t|x',0) = [1 - \nu(x',\theta)\Delta t]\delta(x'-x) + \nu(x',\theta)\Delta t\lambda(|x-x'||\mu'), \tag{6}$$

where x' is the value of the process just before the jump. From the above conditional probability one can derive the master equation by evaluating the time derivative of $p(x,t)$ from the definition and by taking into account all possible values of x and μ before the jump:

$$\frac{\partial}{\partial t}p(x,t) = \lim_{\Delta t\to 0}\left\{\int\int p_{tr}(x,\Delta t|x',0)p(x',t)f(\mu')dx'd\mu' - p(x,t)\right\}\Big/\Delta t. \quad (7)$$

Finally, we obtain the master equation

$$\frac{\partial}{\partial t}p(x,t) = -\nu(x,\theta)p(x,t) + \int\int \nu(x',\theta)\lambda(|x-x'||\mu)f(\mu)p(x',t)dx'd\mu. \quad (8)$$

In the diffusion limit of small wave numbers, which is equivalent to the Kramers-Moyal approximation, the Eq. (8) can be reduced to the fractional equation. The Fourier transform of jump-size distribution $\tilde{\lambda}(k)=\exp(-K^\mu|k|^\mu)$ can then be expanded $\tilde{\lambda}(k)=1-K^\mu|k|^\mu+\cdots$ and the master equation in the Fourier space takes the following form:

$$\frac{\partial\tilde{p}(k,t)}{\partial t} = -\int_1^2 K^\mu|k|^\mu\mathcal{F}[\nu(x,\theta)p(x,t)]f(\mu)d\mu. \quad (9)$$

The inversion of the transform yields the fractional equation

$$\frac{\partial p(x,t)}{\partial t} = \int_1^2 K^\mu\frac{\partial^\mu[\nu(x,\theta)p(x,t)]}{\partial|x|^\mu}f(\mu)d\mu. \quad (10)$$

In order to solve Eq. (10), we approximate $f(\mu)$ by a step function and then express the equation in the form of linear combination of the fractional operators. This form of the solution allows us to account also for processes which involve a discrete spectrum of μ values. The integration interval (1) and (2) is divided into N subintervals of a constant length $\Delta\mu=1/N$. Equation (10), discretized in that way, takes the form

$$\frac{\partial p(x,t)}{\partial t} = \sum_{i=1}^N f_i\frac{\partial^{\mu_i}[\nu(x,\theta)p(x,t)]}{\partial|x|^{\mu_i}}, \quad (11)$$

where $f_i=K^{\mu_i}f(\mu_i)/N$. We want to find a function which satisfies Eq. (11) in the realm of its validity, i.e., for small $|k|$. For a single order parameter $f(\mu_i)=\delta(\mu_i-\mu)$ it is given by the Fox function (2). Similarly, we will try to solve Eq. (11) by assuming the solution in the form of a combination of the Fox functions with unknown coefficients $a_1,\ldots,B_2$:

$$p(x,t) = \mathcal{N}\sum_{i=1}^N p_i, \quad (12)$$

where

$$p_i = a_i(t)H_{2,2}^{1,1}\left[a_i(t)|x|\left|\begin{matrix}(a_1,A_1),(a_2,A_2)\\(b_1,B_1),(b_2,B_2)\end{matrix}\right.\right] \quad (13)$$

and $\mathcal{N}$ is the normalization constant. We insert the Fourier transform of $p(x,t)$ to the Fourier transformed Eq. (11)

$$\dot{\tilde{p}}(k,t) = -\sum_i f_i|k|^{\mu_i}\mathcal{F}[\nu(x,\theta)p(x,t)] \quad (14)$$

and expand both sides of the equation in fractional powers of $|k|$ by utilizing properties of the Fox functions. It is possible to choose the coefficients in Eq. (13) in such a way that all terms of the order different from 0 and μ_i vanish, providing we neglect the higher terms. Only two expansion coefficient are kept: h_{μ_i} on the left-hand side and $h_0^{(i)}$ on the right-hand side. Due to the normalization, each component $\tilde{p}_i$ satisfies $\tilde{p}_i(k,t)-1\sim -h_{\mu_i}(k/a_i)^{\mu_i}$ and it can be expressed in a form which is generic for the symmetric Lévy processes [43]

$$p_i = \frac{1}{\mu_i\sigma_i}H_{2,2}^{1,1}\left[\frac{|x|}{\sigma_i}\left|\begin{matrix}(1-1/\mu,1/\mu),(1/2,1/2)\\(0,1),(1/2,1/2)\end{matrix}\right.\right], \quad (15)$$

where the functions $\sigma_i(t)\sim h_{\mu_i}/a_i^{\mu_i}$ are yet to be determined. Equation (14) takes the form

$$\frac{d}{dt}\left(\sum_j h_{\mu_j}a_j^{-\mu_j}|k|^{\mu_j}+\text{const}\right) = -\sum_j f_j\left(\sum_i a_i^\theta h_0^{(i)}\right)|k|^{\mu_j}. \quad (16)$$

The expansion coefficients are given by [11]

$$h_0^{(i)} = 2\frac{\mu_i+\theta}{2+\theta},$$

$$h_{\mu_i} = -\frac{2}{\pi}(\mu_i+\theta)^2\Gamma(-\mu_i)\Gamma(\mu_i+\theta)\cos(\mu_i\pi/2)\sin\left(\frac{\mu_i+\theta}{2+\theta}\pi\right). \quad (17)$$

By comparing the terms which correspond to consecutive powers of $|k|$ and introducing an auxiliary variable $\xi_i=a_i^{-\mu_i}$, we can transform Eq. (16) to the following system of equations:

$$\frac{h_{\mu_j}}{f_j}\dot{\xi}_j = \sum_i h_0^{(i)}\xi_i^{-\theta/\mu_i} \quad (j=1,\ldots,N). \quad (18)$$

We assume the uniform initial conditions $a_i(0)=\delta(x)/N$; they correspond to the conditions $\xi_i(0)=0$. Then we can express all the functions ξ_i in terms of the ξ_1 by simple integration

$$\xi_i = \frac{h_{\mu_1}f_i}{h_{\mu_i}f_1}\xi_1. \quad (19)$$

In order to evaluate $\xi_1(t)$, one needs to disentangle the following expression:

$$t = \frac{h_{\mu_1}}{f_1}\int_0^{\xi_1}d\xi\Big/\sum_{i=1}^N h_0^{(i)}\left(\frac{h_{\mu_1}f_i}{h_{\mu_i}f_1}\xi\right)^{-\theta/\mu_i}. \quad (20)$$

After the proper normalization, we obtain the formula for the functions σ_i

$$\sigma_i^{\mu_i} = -\frac{\pi}{2}\left[\Gamma(1+\theta)\Gamma\left(-\frac{\theta}{\mu_i+\theta}\right)\sin\left(\frac{\theta}{2+\theta}\pi\right)\right]^{-1} h_{\mu_i}\xi_i \tag{21}$$

and the Eqs. (12) and (15) yield the final solution.

As an example, let us consider the case of uniformly distributed order parameter μ: $f_i=1/N$ for $\mu\in(1.5,2)$ and $f_i=0$ elsewhere. The distribution $p(x,t)$ is determined from Eq. (12) and the Fox functions can be computed by a series expansion for both small and large values of $|x|$:

$$p_i(x,t) = \frac{1}{\pi\sigma_i\mu_i}\sum_{n=0}^{\infty}\frac{\Gamma[1+(2n+1)/\mu_i]}{(2n+1)!!}(-1)^n\left(\frac{x}{\sigma_i}\right)^{2n} \tag{22}$$

and

$$p_i(x,t) = \frac{1}{\pi\sigma_i\mu_i}\sum_{n=1}^{\infty}\frac{\Gamma(1+\mu_i n)}{n!}\sin(\pi\mu_i n/2)\left(\frac{|x|}{\sigma_i}\right)^{-\mu_i n-1}, \tag{23}$$

respectively. The distribution for the intermediate values is difficult to obtain since the series are poorly convergent. The expansion (23) implies that for large $|x|$ the tail approaches the shape $\sim|x|^{-\mu_{\min}-1}$, where $\mu_{\min}$ stands for the smallest value of μ which is involved in the process. The presence of the other μ values shifts this asymptotics to larger $|x|$, compared to the case of the unique μ. On the other hand, one can solve the master equation (8)—which describes the jumping process exactly, without any approximation—by simulating the random walk trajectories by means of the Monte Carlo method. For that purpose, one needs to sample the trapping time from the distribution $w(t)$, the order parameter from $f(\mu)$, and the jump size from $\lambda(x|\mu)$—along each trajectory. The Lévy distributed random numbers can be obtained by using the following algorithm [44]. Let $r_1\in(-\pi/2,\pi/2)$ and $r_2\in(0,\infty)$ are the random numbers which obey the uniform and exponential distribution, respectively. Then the numbers of the form

$$x = \frac{\sin(\mu r_1)}{\cos(r_1)^{1/\mu}}[\cos(r_1-\mu r_1)/r_2]^{1/\mu-1} \tag{24}$$

possess the symmetric Lévy distribution, centered at 0.

The results are presented in Fig. 1 and compared with two limiting distributions which have been calculated with the single values of the order parameter $\mu=1.5$ $[f(\mu)=\delta(\mu-1.5)]$ and $\mu=2$ (the Gaussian case). The upper curve corresponds to $\mu=1.5$; it lies higher because smaller values of μ mean longer, slower decaying tails. The distribution obtained from Eq. (12) agrees with that from the simulations in the limit of large $|x|$ since the fractional equation and the master equation are equivalent there. The curve which corresponds to the distributed order case gradually approaches the shape $\sim|x|^{-2.5}$, the same as for the case $\mu=1.5$.

The transport process for the Lévy flights cannot be regarded as the ordinary diffusion because the second moment of the probability distribution is divergent. To describe the relative transport speed, one can determine a mean characteristic displacement $\langle|x|^\delta\rangle^{1/\delta}$, defined as the moment of order $\delta<\mu_{\min}$, where $\mu_{\min}$ is such that $f(\mu)=0$ for $\mu<\mu_{\min}$. It can be expressed by the Mellin transform from the Fox function $\chi(s)$ and, in the limit $N\to\infty$, reduced to the integral

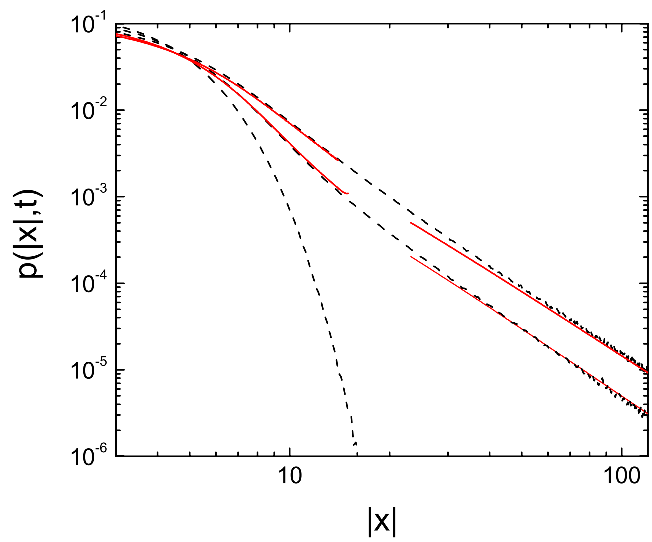

FIG. 1. (Color online) The distribution $p(x,t)$ at $t=5$ obtained from trajectory simulations (dashed lines) for the following cases: $\mu=1.5$ (upper curve), μ uniformly distributed within the interval (1.5,2) (middle curve), and $\mu=2$ (lower curve). The distributions resulting from Eq. (12) are presented by the solid lines. The parameter $\theta=0.4$.

$$\langle|x|^\delta\rangle = 2\int_0^\infty x^\delta p(x,t)dx \sim \sum_i a_i^{-\delta}\chi(-\delta-1)$$
$$\sim \sum_i \sigma_i^\delta\Gamma(1-\delta/\mu_i) \sim \int_1^2 \sigma_\mu^\delta\Gamma(1-\delta/\mu)d\mu, \tag{25}$$

where $\mu=1+i\Delta\mu$. Since, according to Eqs. (19) and (20), all $\xi_i(t)$ rise, $\sigma_i(t)$ rise as well and $\langle|x|^\delta\rangle$ is a monotonically increasing function of time. Figure 2 demonstrates how that function, calculated for $\delta=1$, depends on the interval of μ. Generally, it behaves similar to t^α for large $|x|$. For $\mu\in(1.1,1.5)$ the slope is steep due to the presence of very long jumps. The increase of the slope parameter is especially striking when θ turns to negative. In the case of the Gaussian $\lambda(x)$, negative values of θ correspond to the enhanced diffusion [11].

III. PROCESSES WITH THE DISTRIBUTED WAITING TIME PARAMETER

Let us consider a class of jumping processes with fixed order parameter μ for which the jumping rate parameter θ in Eq. (4) is a random variable, defined by a probability distribution $g(\theta)$. It satisfies the normalization condition $\int_{-1}^{\infty}g(\theta)d\theta=1$. The process is then characterized by the jump-size distribution $\lambda(x)$ and the waiting time distribution $w(t|\theta)$ which determines the time interval between consecutive jumps, conditioned by the parameter θ. The master equation

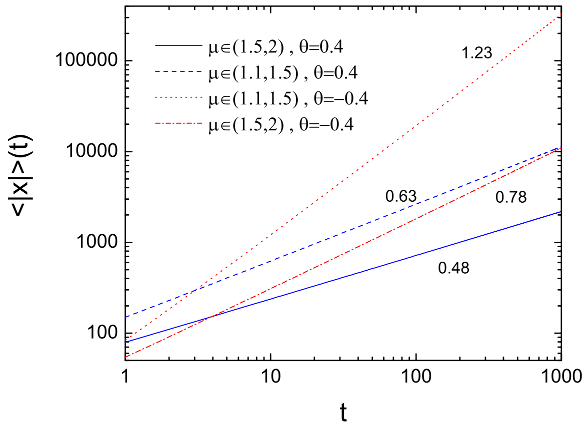

FIG. 2. (Color online) The characteristic displacement $\langle|x|\rangle$ as a function of time for the order parameter μ sampled within two different intervals and for both positive and negative values of θ. The parameter α of the function t^α, which has been fitted to each curve at large t, is indicated in the figure.

for the process can be derived similarly as for the case of the random μ, discussed in Sec. II. The equation reads

$$\frac{\partial}{\partial t}p(x,t) = -\int \nu(x,\theta)g(\theta)p(x,t)d\theta + \int\int \nu(x',\theta)\lambda(|x-x'|) \times g(\theta)p(x',t)dx'd\theta \tag{26}$$

and the corresponding fractional equation in the diffusion limit is of the form

$$\frac{\partial p(x,t)}{\partial t} = K^\mu \int \frac{\partial^\mu[\nu(x,\theta)p(x,t)]}{\partial|x|^\mu}g(\theta)d\theta. \tag{27}$$

The method of solving of Eq. (27) is analogous to that for the random μ. The expressions for the solution and for the mean displacement $\langle|x|^\delta\rangle^{1/\delta}$ are similar—one needs only to substitute $\mu_i\to\mu$, $\theta\to\theta_i$, and $f_i\to g_i$. The asymptotic behavior of $p(x,t)$ can be easily determined. Taking into account the first term in the series (23), we conclude that the slope of the tails depends only on μ: $p(x,t)\sim|x|^{-\mu-1}(|x|\to\infty)$.

The distribution $w(\tau|\theta)$ determines the waiting time interval τ and it depends on x via the jump frequency ν. Let us consider the case of the unique θ. If $\theta>0$ one can substitute τ, in a crude approximation, by its mean value $\bar{\tau}(x)=\langle\tau\rangle_w=1/\nu=|x|^\theta$. The distribution of this variable $w_{\bar{\tau}}[\bar{\tau}(x)]$ is given by the relation $w_{\bar{\tau}}d\bar{\tau}=2p(x,t)dx$, for any time. Knowing how x is distributed for large $|x|$, we conclude that $w_{\bar{\tau}}(\bar{\tau})\sim\bar{\tau}^{-\mu/\theta-1}(\bar{\tau}\to\infty)$ is a power law and the index is determined by the both parameters θ and μ. The resulting non-Markovian process is similar to the standard, decoupled CTRW which used to be described by the fractional equation where the fractional derivative is taken over time and its order equals the slope exponent in the distribution $w(\tau)$ [3]. Therefore, the CTRW with the power-law waiting-time distribution can be regarded as an approximation to processes which are in fact Markovian and possess the x-dependent jumping frequency. In the other words, the non-Markovian property of the decoupled CTRW together with the long tails of $w(\tau)$ can mimic the x-dependence of the waiting time distribution. Taking into account that the exponent in the distribution $w(\tau)$ may not be unique means, in the framework of the decoupled and non-Markovian CTRW, the distributed order of the differentiation over time and it corresponds to the nonunique Hurst exponent [39,41,42]. In the case of the Gaussian process ($\mu=2$), such fractional equation—both for the unique and distributed order—describes subdiffusion.

The problem of jumping process with the distributed parameter θ is closely related to that which involves the jumping frequency ν of the x depedence other than algebraic. Indeed, we rewrite Eq. (26) in the form

$$\frac{\partial}{\partial t}p(x,t) = -\nu'(x)p(x,t) + \int \nu'(x')\lambda(|x-x'|)p(x',t)dx', \tag{28}$$

i.e., as the ordinary master equation for the unique θ and μ, with a new frequency $\nu'(x)=\int\nu(x,\theta)g(\theta)d\theta$. If we assume that $g(\theta)=0$ for $\theta<0$, ν' can be expressed in the form of the Laplace transform

$$\nu'(s) = \int_0^\infty g(\theta)e^{-s\theta}d\theta, \tag{29}$$

where $s=\ln|x|$. Therefore, the solution of the fractional equation (1) for a given $\nu[\ln(|x|)]$ is possible by utilizing the solution for the problem of the distributed θ; it requires the inversion of the Laplace transform (29). The existence condition restricts the applicability of the presented method to a rather weak x dependence of the frequency. Moreover, the original function $g(\theta)$ should be normalizable. The latter condition, however, can be dropped if only large values of $|x|$ are interesting, that usually is the case; then we can impose an upper cut on the integration interval.

Let us consider an example of the jumping rate $\nu(x)$ which decreases logarithmically for $|x|\to\infty$:

$$\nu(x) = \frac{\gamma}{\gamma+\ln|x|}\frac{1-e^{-\gamma g}|x|^{-g}}{1-e^{-\gamma g}}, \tag{30}$$

where $\gamma,g>0$. This case can be solved exactly and directly compared with the simulations. Inversion of the Laplace transform yields

$$g(\theta) = \frac{\gamma}{1-e^{-\gamma g}}e^{-\gamma\theta} \quad \text{for } \theta\in(0,g). \tag{31}$$

Figure 3 presents the distribution $p(x,t)$ which is the solution of Eq. (27) with $g(\theta)$ given by Eq. (31); the result of the Monte Carlo trajectory simulations is also shown. The distributions are compared with the case of the constant ν. The tail of the distribution for the former process lies slightly lower. That difference stems from the fact that for the logarithmic $\nu(x)$ all values of the parameter $\theta\in(0,1)$ are involved, whereas the case ν=const corresponds solely to $\theta=0$; positive values of θ hamper the expansion of the distribution. On the other hand, the slope is the same for both cases since it depends only on μ.

Up to now, we have considered the processes for which the distributions w and λ are coupled via the x variable. In some physical situations, however, the trapping may influ-

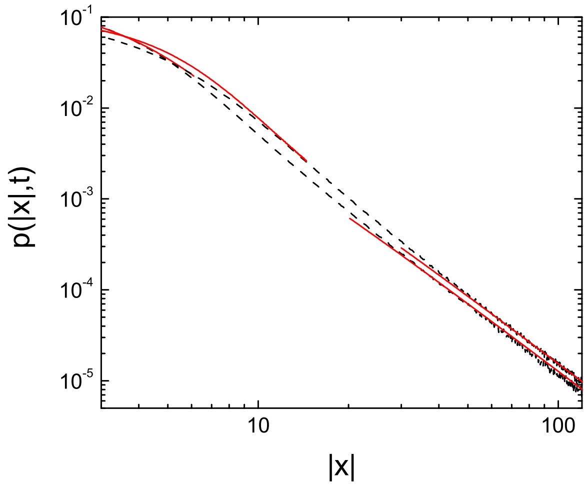

FIG. 3. (Color online) Lower curves: the distribution $p(x,t)$ obtained from trajectory simulations for the jumping rate (30) with parameters $\gamma=1$ and $g=1$ (dashed line) and as the solution of Eq. (27), expanded at both small and large $|x|$ (solid lines). Upper curves: the same but for the case $\nu=1$. For the both processes $t=5$ and $\mu=1.5$.

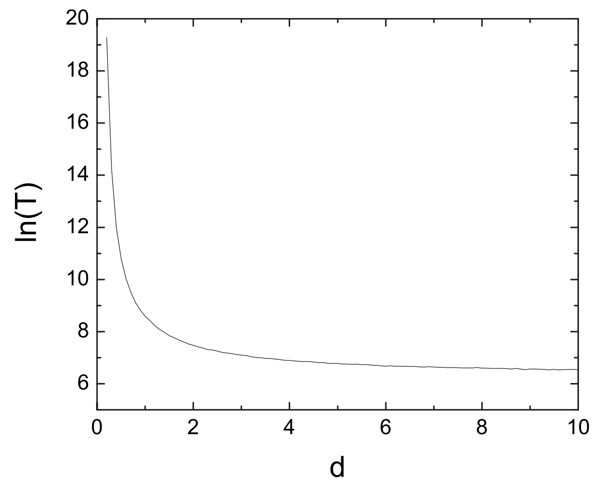

FIG. 4. Mean first passage time for the process defined by Eq. (32) for $\theta(\mu)$ given by Eq. (33). The absorbing barrier is positioned at $L=50$, $\mu_0=1.5$.

ence the jumping mechanism and vice versa, just on the level of individual trajectories. In the framework of the present formalism, such a coupling requires the introduction of a direct dependence between the parameters $\theta=\theta(\mu)$. If we assume that the order parameter μ is a stochastic variable distributed according to $f(\mu)$, the master equation for that process reads

$$\frac{\partial}{\partial t}p(x,t) = -\int \nu[x,\theta(\mu)]f(\mu)p(x,t)d\mu + \int\int \nu[x',\theta(\mu)] \times \lambda(|x-x'|)f(\mu)p(x',t)dx'd\mu. \quad (32)$$

The form of the function $\theta(\mu)$ must be imposed by physical conditions which are specific to a concrete problem. Let us consider only a simple example

$$\theta(\mu) = \frac{1}{\mu-\mu_0+d}, \quad (33)$$

where $\mu \in (\mu_0,2)$ and $d>0$. Such dependence means that if the probability of a long jump is large (small μ), it will be compensated by the increased probability of large waiting time. The strength of that enhanced trapping is governed by the parameter d. We wish to calculate the mean first passage time T, i.e., the average time a trajectory needs to reach a given distance L, regarded as an absorbing barrier. For that purpose one needs to solve the Eq. (32) with the boundary condition $p(L,t)=0$. The result of the numerical calculations is presented in Fig. 4: T rises to infinity when $d\to 0$. For the Lévy flights with any but unique choice of the parameters μ and θ, the time T is always finite [45].

IV. CONCLUSIONS

We have discussed the random walk process which is defined by the Lévy distribution of jump size and by the Poissonian waiting time distribution. The nonhomogeneity of the medium is included in the model by allowing for the dependence of the jumping rate on the process value. The process, in the diffusion limit, is governed by the Lévy distribution and it is characterized by a simple scaling. Our purpose was to study the problem of multiscaling which emerge when the parameters of the process—either the Lévy distribution order parameter μ or the jumping rate parameter θ—are not unique but regarded as random variables with given probability density distributions. We have derived the master equations and the corresponding fractional equations. The solution of the latter ones can be expressed as a combination of the Fox functions—each of them corresponds to a single Lévy process with a simple scaling.

We have restricted our considerations mainly to the jumping rate in the scaling form $\nu(x)=|x|^{-\theta}$. However, the process with the distributed parameter θ appears to be equivalent to the case of a process with a more general form of $\nu(x)$; if the inverted Laplace transform of the function $\nu[\ln(|x|)]$ exists, which is the case for a rather slow decrease of the jumping rate with $|x|$, the solution of the corresponding fractional equation can then be achieved and it has the form of a linear combination of the Fox functions.

All the processes in which the jumping size obeys the Lévy distribution, both for the unique and distributed model parameters, are expected to possess the divergent second moment. Then the diffusion must be accelerated and the first passage time must be finite. In particular, the μ dependence of the tail of $p(x,t)$ for the case of the distributed θ is not sensitive on $g(\theta)$; it coincides with that for the Lévy process which corresponds to a unique θ. However, the transport can be slowed down considerably if one allows for an explicit dependence between the parameters θ and μ, i.e., between temporal and spatial ingredients of the trajectory evolution. We have demonstrated that if penalizing long jumps by large values of θ is sufficiently strong, the mean first passage time becomes infinite.

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